Fuzzy Inference

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Summary

- Introduction
- Fuzzy variables
- Fuzzy implication
- Fuzzy composition and inference

Introduction

- In order to discuss about a phenomenon from the real world it is necessary to use a number fuzzy sets
- For example, consider *room temperature*, one could use *low, medium and high temperature*
- Note that these sets may overlap, allowing some temperatures belong partially to more than one set

Introduction cont.

- The process includes the definition of the membership functions
- The Universe of discourse is also an important parameter

Fuzzy Variables

Fuzzy Variable

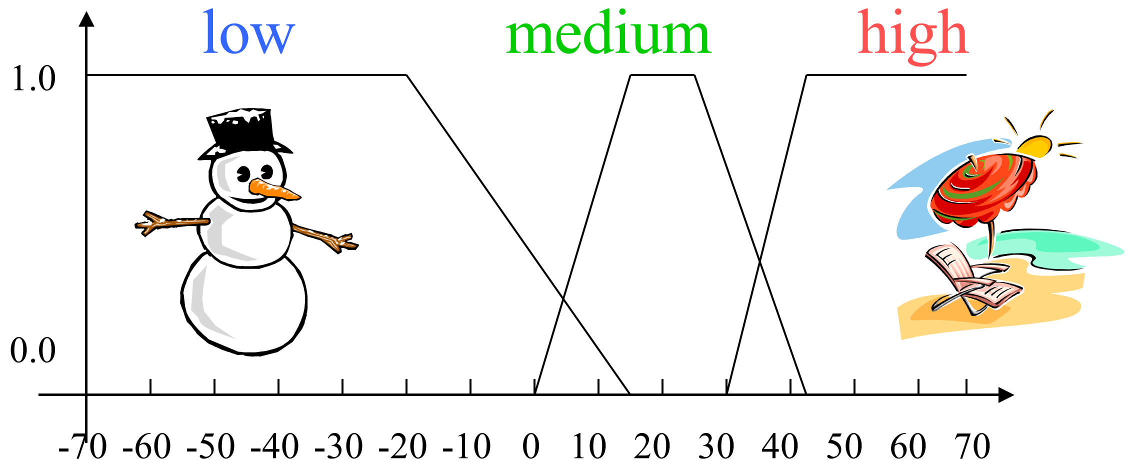
- A fuzzy variable is defined by the quadruple

$$V = \{ x, l, u, m\}$$

- *X* is the variable symbolic name: *temperature*
- *L* is the set of labels: *low, medium and high*
- *U* is the **universe of discourse**
- *M* are the semantic rules that define the meaning of each label in *L* (membership functions).

Fuzzy Variable Example

- X = Temperature
- L = {low, medium, high}
- U = {x∈X | -70º <= x <= +70º}
- M =

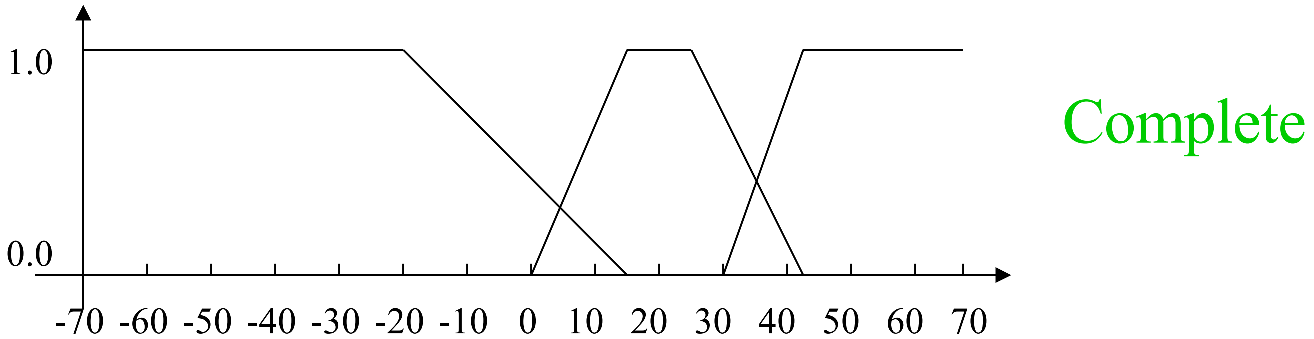


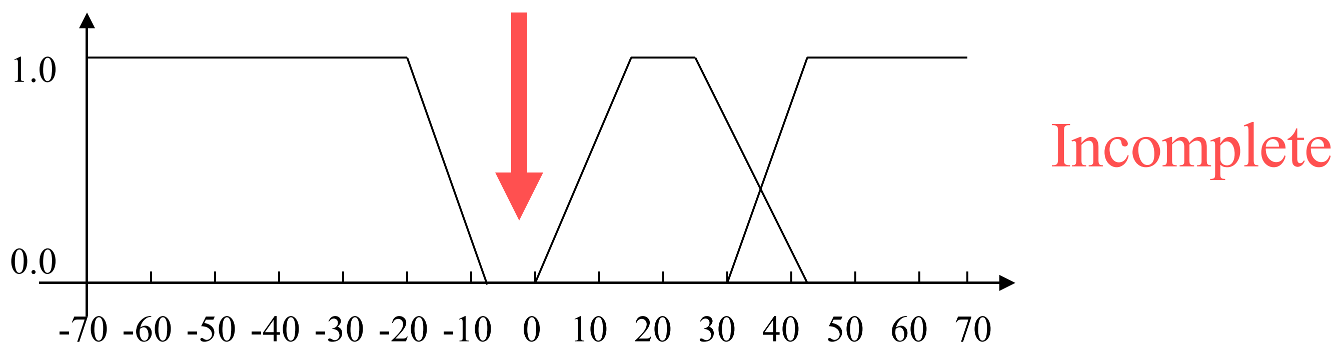
Membership Functions?

- Subjective evaluation: The shape of the functions is defined by specialists
- Ad-hoc: choose a simple function that is suitable to solve the problem
- Distributions, probabilities: information extracted from measurements
- Adaptation: testing
- Automatic: algorithms used to define functions from data

Variable Terminology

- Completude: A variable is complete if for any $x \in X$ there is a fuzzy set such as $\mu(x)>0$





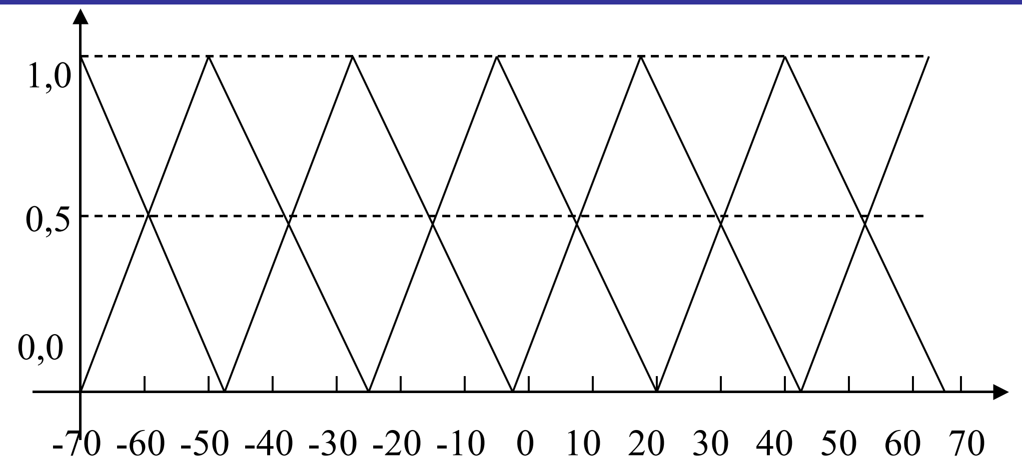
Partition of Unity

- A fuzzy variable forms a partition of unity if for each input value *x*

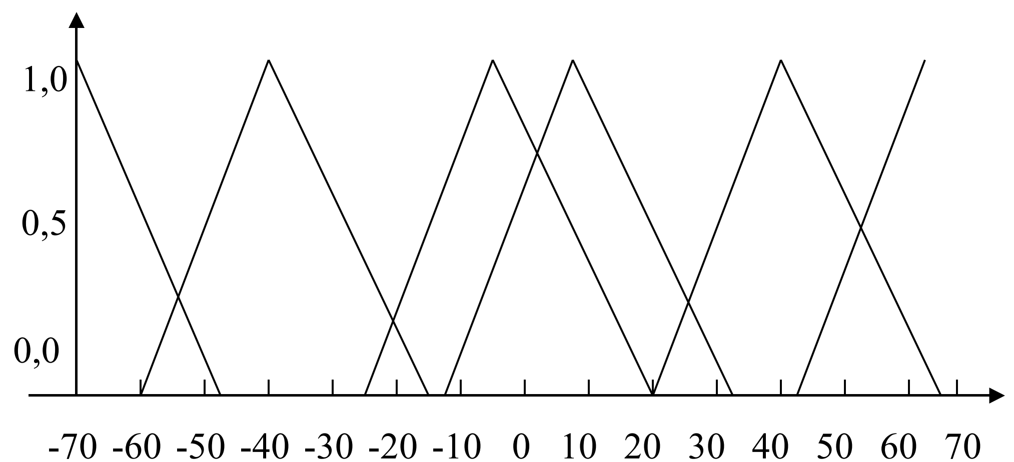
$$\sum_{i=1}^{p} \mu_{A_i}(x) \equiv 1$$

- where *p* is the number of sets to which *x* belongs
- There is no rule to define the overlapping degree between two neighbouring sets
- A rule of thumb is to use 25% to 50%

Partition of Unity



Partition of Unity



No Partition of Unity

Partition of Unity cont

- Any complete fuzzy variable may be transformed into a partition of unity using the equation

$$\mu_{\hat{A}_i}(x) = \frac{\mu_{A_i}(x)}{\sum_{j=1}^{p} \mu_{A_j}(x)} \quad for \; i = 1, \ldots, p$$

Implications

Implications

- If $x \in A$ then $y \in B$.
- P is a proposition described by the set A
- Q is a proposition described by the set B
- $P \rightarrow Q$: If $x \in A$ then $y \in B$

Implications

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$
f	f	t	f	f	t
f	t	t	f	t	t
t	f	f	f	t	f
t	t	f	t	t	t

- Implication is the base of fuzzy rules
- $a \rightarrow b = \neg a \vee b$

Implication as a Relation

- The rule *if x is A then y is B* can be described as a relation

$$R(x,y) = \sum_{x_i,y_i} \mu(x_i,y_i)/(x_i,y_i)$$

$$R(x,y) = \int_{x_i,y_i} \mu(x_i,y_i)/(x_i,y_i)$$

- where μ(x,y) is the relation we want to discover, for example $\neg x \vee y$
- There are over 40 implication relations reported in the literature

Interpretations of Implications

- There are two ways of interpreting implication
- $p \rightarrow q$: meaning p is coupled to q and implication is a T-norm operator

p is coupled with q

- Commonly used T-norms are:
- Mandami $$R(x_i, y_i) = \sum_{x_i, y_i} \mu_A(x_i) \wedge \mu_B(y_i) / (x_i, y_i)$$
- Larson $$R(x_i, y_i) = \sum_{x_i, y_i} \mu_A(x_i) \times \mu_B(y_i) / (x_i, y_i)$$
- Bounded Difference

$$R(x_i, y_i) = \sum_{x_i, y_i} 0 \vee (\mu_A(x_i) + \mu_B(y_i) - 1) / (x_i, y_i)$$

p entails q

- These implication operators are generalisations of the material implications in two-valued logic as in
- a → b = ¬a ∨ b

$$R(x_i, y_i) = \sum_{x_i, y_i} (1 - \mu_A(x_i)) \vee \mu_B(y_i) / (x_i, y_i)$$

- a → b = ¬a ∨ (a ∧ b)

$$R(x_i, y_i) = \sum_{x_i, y_i} (1 - \mu_A(x_i)) \vee (\mu_A(x_i) \wedge \mu_B(y_i)) / (x_i, y_i)$$

P entails q cont

- Goguen (1969)

$$R(x,y)=\begin{cases} 1 & \text{if } \mu_A(x)\leq\mu_B(x) \\ \mu_A(x)/\mu_B(x) & \text{if } \mu_A(x)>\mu_B(x) \end{cases}$$

- Kurt Godel

$$R(x,y)=\begin{cases} 1 & \text{if } \mu_A(x)\leq\mu_B(x) \\ \mu_B(x) & \text{if } \mu_A(x)>\mu_B(x) \end{cases}$$

Inference

Inference

- Fuzzy inference refers to computational procedures used for evaluating fuzzy rules of the form if x is A then y is B
- There are two important inferencing procedures
  - Generalized modus ponens (GMP) - mode that affirms
  - Generalized modus tollens (GMT) – mode that denies

Modus Ponens

- If *x* is *A* then *y* is *B*
- We know that x is A' then we can infer that y is B'
- All men are mortal (rule)
- *Socrates* is a *man* (this is true)
- So *Socrates* is *mortal* (as a consequence)
- (A and (A -> B)) -> B

Fuzzy Modus Ponens

- If x is A then y is B
- We know that x is A’ then we can infer that y is B’
- *Tall* men are *heavy* (rule)
- *John* is *tall* (this is true)
- So *John* is *heavy* (as a consequence)
- (A and (A -> B)) -> B

Fuzzy Modus Ponens proof

$$(A\wedge(A\rightarrow B))\rightarrow B \quad start$$

$$(A\wedge(\overline{A}\vee B))\rightarrow B \quad implication$$

$$(A\wedge\overline{A})\vee(A\wedge B))\rightarrow B \quad distributivity$$

$$(\emptyset\vee(A\wedge B))\rightarrow B$$

$$A\wedge B\rightarrow B$$

$$(\overline{A\wedge B})\vee B \quad implication$$

$$(\overline{A}\vee\overline{B})\vee B \quad De\,Morgan$$

$$(\overline{A}\vee(\overline{B}\vee B)) \quad Associativity$$

$$(\overline{A}\vee X)$$

$$X$$

Modus Tollens

- If *x* is *A* then *y* is *B*
- We know that y is not B then we can infer that x is not A
- *All murderers owns axes* (rule)
- *John* does *not own an axe* (this is true)
- So *John* is *not a murderer* (as a consequence)
- (not B and (A -> B)) -> not A

Fuzzy Modus Tollens

- If *x* is *A* then *y* is *B*
- We know that y is not B then we can infer that x is not A
- All rainy days are cloudy (rule)
- *Today* is *not cloudy* (this is true)
- So *Today* is *not raining* (as a consequence)
- (not B and (A -> B)) -> not A

Fuzzy Modus Tollens proof ?

Reasoning Methods

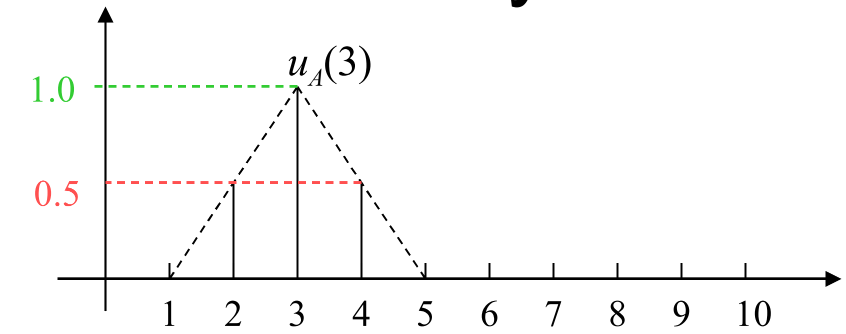
- Backward Chaining: the reasoning engine is presented with a goal and asked to find all the relevant, supporting processes that lead to this goal.
- Forward Chaining: data is collected and and a sustainable problem state and, eventually a solution state is built.
- Fuzzy Reasoning: rules are run in parallel. Every rule contributes to the final shape of the consequent solution. When all rules are evaluated the resulting fuzzy sets are defuzzified.

How to find the consequent

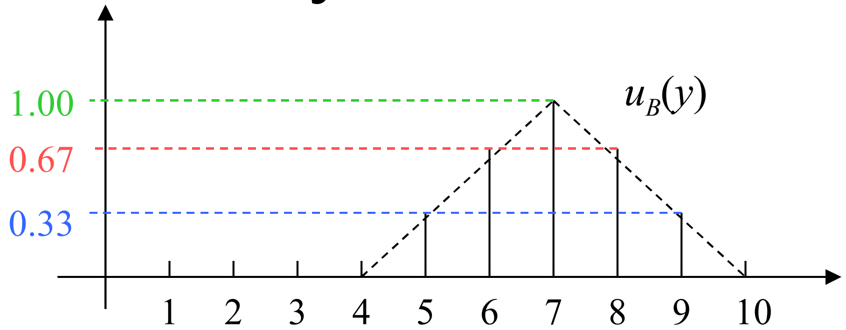
- If *x* is *A* then *y* is *B*
- If *x is A'*, we want to know whether *y is B'*
- This rule is an implication *R(x,y)*
- In order to compute this rule we need to establish *R(x,y)*

Example

- Consider the fuzzy set *A*



- and the fuzzy set *B*



Example 1

$$A = \sum_{i=0}^{10} \mu_A(x_i)/x_i = 0.5/2 + 1.0/3 + 0.5/4$$

$$B = \sum_{i=0}^{10} \mu_B(y_i)/y_i = 0.33/5 + 0.67/6 + 1.0/7 + 0.67/8 + 0.33/9$$

Example 2

- We will use the Mamdani implication function

$$\mu(x_i, y_i) = \mu_A(x_i) \wedge \mu_B(y_i)$$

$$R(x_i, y_i) = \sum_{(x_i, y_i)} \mu(x_i, y_i) / (x_i, y_i)$$

$$R(x_i, y_i) = 0.33/(2,5) + 0.5/(2,6) + 0.5/(2,7) +$$
$$0.5/(2,8) + 0.33/(2,9) + 0.33/(3,5) + 0.67/(3,6) +$$
$$1.0/(3,7) + 0.67/(3,8) + 0.33/(3,9) + 0.33/(4,5) +$$
$$0.5/(4,6) + 0.5/(4,7) + 0.5/(4,8) + 0.33/(4,9)$$

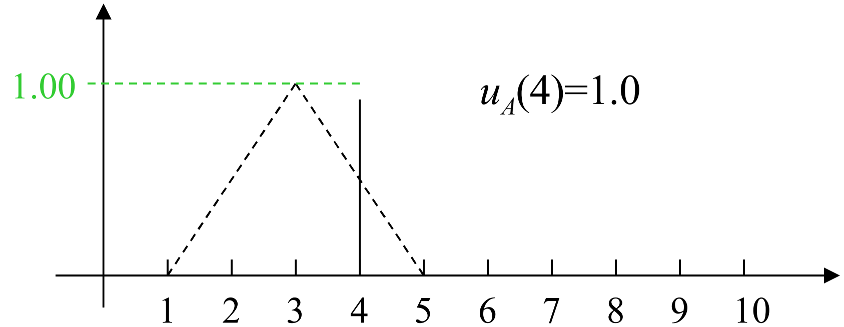
Example 3

$$R(x_i, y_i) = \sum_{(x_i, y_i)} \mu(x_i, y_i) / (x_i, y_i)$$

A \ B	5	6	7	8	9
2	0,33	0,50	0,50	0,50	0,33
3	0,33	0,67	1,00	0,66	0,33
4	0,33	0,50	0,50	0,50	0,33

Example 4

- Consider the rule if x is A then y is B
- Consider the statement x is A', what is the conclusion?



$$A' = \sum_{i=0}^{10} \mu_A(x_i)/x_i = 1.0/4$$

Example

- $B'(y_i)=A'(x_i)oR(x_i,y_i)$

$$B'(y_i)=[0\ 0\ 1]\circ\begin{bmatrix}0.33 & 0.50 & 0.50 & 0.50 & 0.33\\0.33 & 0.67 & 1.00 & 0.66 & 0.33\\0.33 & 0.50 & 0.50 & 0.50 & 0.33\end{bmatrix}$$

$$B'=0.33/5+0.50/6+0.50/7+0.50/8+0.33/9$$

Example